Consideration of end-fed antenna

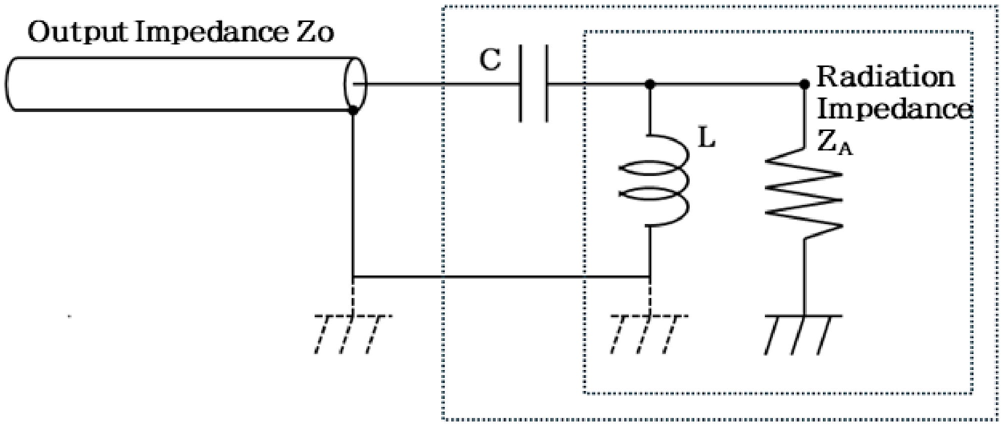
Attachment 1

How to calculate constants for impedance matching LC circuit (memorandum)

When transmitter output impedance Z_o, radiation resistance Z_A, set frequency f_0, and set angular frequency $\omega_0 = 2\pi f_0$, the constants of the impedance matching circuit in the figure below are as follows.

$$L = \frac{Z_A}{\omega_0}\sqrt{\frac{Z_o}{Z_A - Z_o}}$$

$$C = \frac{1}{\omega_0\sqrt{Z_o(Z_A - Z_o)}}$$



At this time, the resonant angular frequency $\omega_R = \frac{1}{\sqrt{LC}}$ and resonant frequency $f_R = \frac{\omega_R}{2\pi}$ of this LC circuit is as follows:

$$LC = \frac{Z_A}{\omega_0}\sqrt{\frac{Z_o}{Z_A - Z_o}}\frac{1}{\omega_o\sqrt{Z_o(Z_A - Z_o)}} = \frac{Z_A}{\omega_o^2(Z_A - Z_o)}$$

Therefore,

$$\omega_R = \frac{1}{\sqrt{LC}} = \omega_o\sqrt{\frac{Z_A - Z_o}{Z_A}} \qquad f_R = \frac{\omega_R}{2\pi}$$

The LC resonance frequency will be slightly lower than the set frequency.

The combined impedance Z_{LZA} of the parallel circuit of coil L and radiation resistance Z_A is

$$\begin{aligned} Z_{LZA} &= \frac{Z_A j\omega_0 L}{Z_A + j\omega_0 L} \\ &= \frac{Z_A j\omega_0 L}{Z_A + j\omega_0 L}\ \frac{Z_A - j\omega_0 L}{Z_A - j\omega_0 L} \\ &= \frac{\omega_0^2 L^2 Z_A + j\omega_0 L Z_A^2}{Z_A^2 + \omega^2 L^2} \end{aligned}$$

The overall impedance including C is

$$\begin{aligned} Z &= \frac{1}{j\omega_0 C} \quad + \frac{\omega_0^2 L^2 Z_A + j\omega_0 L Z_A^2}{Z_A^2 + \omega^2 L^2} \\ &= \frac{\omega_0^2 L^2 Z_A}{Z_A^2 + \omega^2 L^2} + j\ \left(\frac{\omega_0 L Z_A^2}{Z_A^2 + \omega^2 L^2} - \frac{1}{\omega_0 C}\right) \end{aligned}$$

In order to set the reactance component to 0 at the set angular frequency ω_0,

$$\frac{\omega_0 L Z_A^2}{Z_A^2+\omega_0^2 L^2} - \frac{1}{\omega_0 C} = 0$$

$$\frac{\omega_0 L Z_A^2}{Z_A^2+\omega_0^2 L^2} = \frac{1}{\omega_0 C}$$

$$Z_A^2 + \omega_0^2 L^2 = \omega_0^2 LC Z_A^2$$

At this time, $Re(Z) = Z_o$

$$\frac{\omega_0^2 L^2 Z_A}{Z_A^2+\omega_0^2 L^2} = Z_o$$

$$\omega_0^2 L^2 Z_A = Z_o\left(Z_A^2 + \omega_0^2 L^2\right)$$

$$L^2 = \frac{Z_o Z_A^2}{\omega_0^2 (Z_A - Z_o)}$$

$$L = \frac{Z_A}{\omega_0}\sqrt{\frac{Z_o}{Z_A - Z_o}}$$

Similarly for C

$$Z_A^2 + \omega_0^2 L^2 = \omega_0^2 LC Z_A^2 \qquad \frac{\omega_0^2 L^2 Z_A}{Z_A^2+\omega_0^2 L^2} = Z_o$$ から、

$$\frac{L}{C Z_A} = Z_o$$

$$C = \frac{1}{Z_A Z_o}\frac{Z_A}{\omega_0}\sqrt{\frac{Z_o}{Z_A - Z_o}} = \frac{1}{\omega_0}\frac{1}{\sqrt{Z_o(Z_A - Z_o)}}$$

Resonant frequency f_R between L and C

$$\begin{aligned} LC &= \frac{Z_A}{\omega_0}\sqrt{\frac{Z_o}{Z_A - Z_o}}\frac{1}{\omega_0}\frac{1}{\sqrt{Z_o(Z_A - Z_o)}} \\ &= \frac{1}{\omega_0^2}\frac{Z_A}{ZA - Z_o} \end{aligned}$$

$$\begin{aligned} f_R &= \frac{1}{2\pi\sqrt{LC}} \\ &= \frac{1}{\frac{2\pi}{\omega_0}\sqrt{\frac{Z_A}{ZA - Z_o}}} \\ &= \frac{\omega_0}{2\pi}\sqrt{\frac{Z_A - Z_o}{Z_A}} \\ &= f_0\sqrt{\frac{Z_A - Z_o}{Z_A}} \end{aligned}$$

The resonant frequency f_R of L and C will be set to $\sqrt{\frac{Z_A - Z_o}{Z_A}}$ times the set frequency f_0.

Numerical examples

$f_0 = 7.10 MHz$

$$\begin{array}{llll} Z_A = 2500k\Omega & L = 8.006\mu H & C = 64.05pF & f_R = 7.029MHz \\ Z_A = 3000k\Omega & L = 8.775\mu H & C = 58.37pF & f_R = 7.041MHz \\ Z_A = 3500k\Omega & L = 9.445\mu H & C = 53.97pF & f_R = 7.049MHz \end{array}$$

$f_0 = 21.3 MHz$

$$\begin{array}{llll} Z_A = 2500k\Omega & L = 2.669\mu H & C = 21.35pF & f_R = 21.09MHz \\ Z_A = 3000k\Omega & L = 2.918\mu H & C = 19.46pF & f_R = 21.12MHz \\ Z_A = 3500k\Omega & L = 3.148\mu H & C = 17.99pF & f_R = 21.15MHz \end{array}$$

$f_0 = 50.2MHz$

$Z_A = 2500k\Omega$	$L = 1.132\mu H$	$C = 9.058pF$	$f_R = 49.70MHz$
$Z_A = 3000k\Omega$	$L = 1.238\mu H$	$C = 8.255pF$	$f_R = 49.78MHz$
$Z_A = 3500k\Omega$	$L = 1.336\mu H$	$C = 7.633pF$	$f_R = 49.84MHz$

$f_0 = 145MHz$

$Z_A = 2500k\Omega$	$L = 0.392\mu H$	$C = 3.136pF$	$f_R = 143.54MHz$
$Z_A = 3000k\Omega$	$L = 0.429\mu H$	$C = 2.868pF$	$f_R = 143.79MHz$
$Z_A = 3500k\Omega$	$L = 0.462\mu H$	$C = 2.643pF$	$f_R = 143.96MHz$

Furthermore, if we set $Z_A = R_A + jX_A$

Substitute $Z_A = R_A + jX_A$ into the formula of Z_{LZA}.

$$\begin{aligned}
Z_{LZA} &= \frac{(R_A + jX_A)j\omega_0 L}{(R_A + jX_A) + j\omega_0 L} \\
&= \frac{-\omega_0 LX_A + j\omega_0 LR_A}{R_A + j(\omega_0 L + X_A)} \frac{R_A - j(\omega_0 L + X_A)}{R_A - j(\omega_0 L + X_A)} \\
&= \frac{-\omega_0 LX_A R_A + \omega_0 LR_A(\omega_0 L + X_A) + j(\omega_0 LX_A(\omega_0 L + X_A) + \omega_0 LR_A^2)}{R_A^2 + (\omega_0 L + X_A)^2} \\
&= \frac{-\omega_0^2 L^2 R_A + j\omega_0 L(\omega_0 LX_A + X_A^2 + R_A^2)}{R_A^2 + (\omega_0 L + X_A)^2}
\end{aligned}$$

The overall impedance including C is

$$Z = \frac{1}{j\omega_0 C} + \frac{-\omega_0^2 L^2 R_A + j\omega_0 L(\omega_0 LX_A + X_A^2 + R_A^2)}{R_A^2 + (\omega_0 L + X_A)^2}$$

The real part and imaginary part are

$$Re(Z) = \frac{\omega_0^2 L^2 R_A}{R_A^2 + (\omega L + X_A)^2} \quad Im(Z) = \frac{\omega_0 L(\omega_0 LX_A + X_A^2 + R_A^2)}{R_A^2 + (\omega_0 L + X_A)^2} - \frac{1}{\omega_0 C}$$

Since the output impedance Z_o of the transmitter is $Z_o = R_o + j0$,

$$Re(Z) = \frac{\omega_0^2 L^2 R_A}{R_A^2 + (\omega L + X_A)^2} = R_o \quad Im(Z) = \frac{\omega_0 L(\omega_0 LX_A + X_A^2 + R_A^2)}{R_A^2 + (\omega_0 L + X_A)^2} - \frac{1}{\omega_0 C} = 0$$

From the formula $Re(Z)$,

$$\omega_0^2 L^2 R_A = R_o R_A^2 + R_o(\omega_0^2 L^2 + 2\omega_0 LX_A + X_A^2)$$

$$\omega_0^2 L^2 R_A^2 - R_o R_A^2 - \omega_0^2 L^2 R_o - 2\omega_0 LX_A R_o - X_A^2 R_o = 0$$

$$\omega_0^2 (R_A - R_o)L^2 - 2\omega_0 X_A R_o L - R_o(R_A^2 + X_A^2) = 0$$

Since it is a quadratic equation for L,

$$\begin{aligned}
L &= \frac{\omega_0 X_A R_A + \sqrt{(\omega_0 X_A R_o)^2 + \omega_0^2 (R_A - R_o)R_o(R_A^2 + X_A^2)}}{\omega_0^2 (R_A - R_o)} \\
&= \frac{\omega_0 X_A R_A + \omega_0 \sqrt{R_A R_o(R_A^2 + X_A^2 - R_A R_o)}}{\omega_0^2 (R_A - R_o)} \\
&= \frac{X_A R_A + \sqrt{R_A R_o(R_A^2 + X_A^2 - R_A R_o)}}{\omega_0 (R_A - R_o)}
\end{aligned}$$

From the formula of $Im(Z)$,

$$\omega_0^2 LC(\omega_0 LX_A + X_A^2 + R_A^2) = R_A^2 + (\omega_0 L + +X_A)^2$$

$$C = \frac{R_A^2 + (\omega_0 L + +X_A)^2}{\omega_0^2 L(\omega_0 L X_A + X_A^2 + R_A^2)}$$

Check when $X_A = 0$

$$\begin{aligned}
L &= \frac{\sqrt{R_A R_o (R_A^2 - R_A R_o)}}{\omega_0 (R_A - R_o)} \\
&= \frac{R_A \sqrt{(R_o (R_A - R_o)}}{\omega_0 (R_A - R_o)} \\
&= \frac{R_A}{\omega_0} \sqrt{\frac{R_o}{R_A - R_o}}
\end{aligned}$$

$$\begin{aligned}
C &= \frac{R_A^2 + \omega_0^2 L^2}{\omega_0^2 L R_A^2} \\
&= \frac{R_A^2 + \omega_0^2 \frac{R_A^2}{\omega_0^2} \frac{R_o}{R_A - R_o}}{\omega_0^2 \frac{R_A}{\omega_0} \sqrt{\frac{R_o}{R_A - R_o}} R_A^2} \\
&= \frac{1 + \sqrt{\frac{R_o}{R_A - R_o}}}{\omega_0 R_A} \\
&= \frac{R_A - R_o + R_o}{\omega_0 R_A \sqrt{R_o (R_A - R_o)}} \\
&= \frac{1}{\omega_0 \sqrt{R_o (R_A - R_o)}}
\end{aligned}$$

that's all